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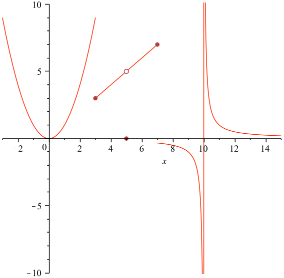
- The discontinuities may be removable discontinuities, jump discontinuities or infinite discontinuities.

Continuous from left and right

All of the above-mentioned types of discontinuities are evident in the graph below:



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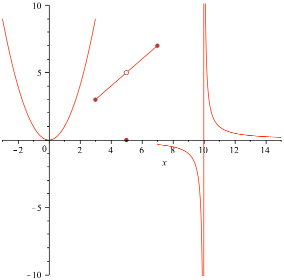

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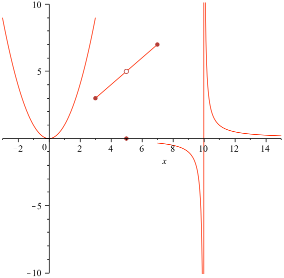

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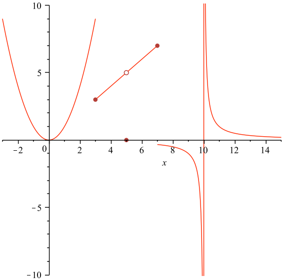

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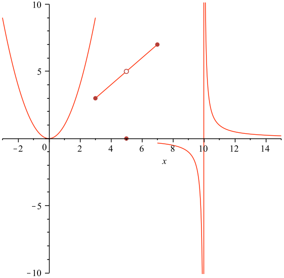

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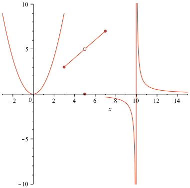

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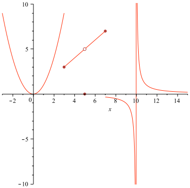

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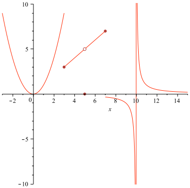

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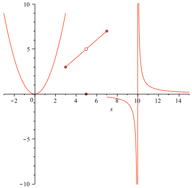

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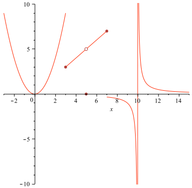

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- **Theorem (proof at end of notes)** The functions $\sin x$ and $\cos x$ are continuous on the interval $(-\infty, \infty)$. In particular; for any real number a, we can evaluate the limits below by direct substitution

$$\lim_{x \to a} \sin x = \sin a, \qquad \lim_{x \to a} \cos x = \cos a.$$

Algebraic combinations of the above functions

Theorem 2 If f and g are continuous at a and c is constant, then the following functions are also continuous at a:

1. $f + g$
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- For example the function $k(x) = \sqrt[3]{x}(x^2 + 2x + 1) + \frac{x+1}{x-10}$ is continuous on its domain. (see notes for details).

Examples

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$$m(x) = \begin{cases} cx^2 + 1 & x \geq 2 \\ 10 - x & x < 2 \end{cases}$$

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- That is when $4c + 1 = 8$ or $4c = 7$ or $c = 7/4$.

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Using continuity to calculate limits.

Note If a function $f(x)$ is continuous on its domain and if a is in the domain of f, then

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That is, if a is in the domain of f, we can calculate the limit at a by evaluation. If a is not in the domain of f, we can sometimes use the methods discussed in the last lecture to determine if the limit exists or find its value.

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Composition of Functions

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We can further expand our catalogue of function continuous on their domains by considering composition of functions.

Theorem 3 If G is a continuous at a and F is continuous at $G(a)$, then the composite function $F \circ G$ given by $(F \circ G)(x) = F(G(x))$ is continuous at a, and

$$\lim_{x \longrightarrow a} (F \circ G)(x) = (F \circ G)(a).$$

That is :

$$\lim_{x \longrightarrow a} F(G(x)) = F(\lim_{x \longrightarrow a} G(x)).$$

Note that when the above conditions are met, we can calculate the limit by direct substitution.

Recall that the domain of $F \circ G$ is the set of points $\{x \in \mathrm{dom} G | G(x) \in \mathrm{dom} F\}$. Using this and the theorem above we get:

Theorem If $f(x) = F(G(x))$, then f is continuous at all points in its domain if G is continuous at all points in its domain and F is continuous at all points in its domain. (Note that we can repeat the process to get the same result for a function of the form $F(G(H(x)))$.)

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Example (a) Find the domain of the following function and determine if it is continuous on its domain?:

$$f(x) = \cos(x^3 + 1).$$

Recall : If $G(x) = x^3 + 1$ and $F(x) = \cos x$, then $F(G(x)) = \cos(x^3 + 1)$.

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- $\lim_{x \to 5} \cos(x^3 + 1) = \cos(5^3 + 1) = \cos(126)$.

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Example Recall that last day we found $\lim_{x \to 0} x^2 \sin(1/x)$ using the squeeze theorem. What is the limit?

Does the function

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- yes if we define $n_1(x) = 0$ at $x = 0$, the function n_1 will be continuous. (note that $\sin(1/x)$ is continuous on its domain because it is a composition of continuous functions and hence $x^2 \sin(1/x)$ is also continuous on its domain ($\{x | x \neq 0\}$).

Intermediate Value Theorem

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Slopes of Tangents/Curves

Recall from Lecture 2 that our intuition tells us that the slope of the tangent line to the curve at the point P is (see Mathematica Files)

$$m = \lim_{Q \to P} m_{PQ} = \lim_{x \to a} m_{PQ} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

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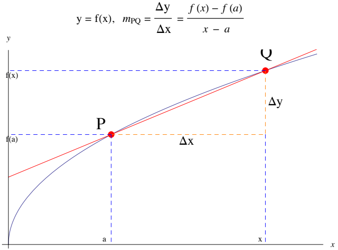
Example



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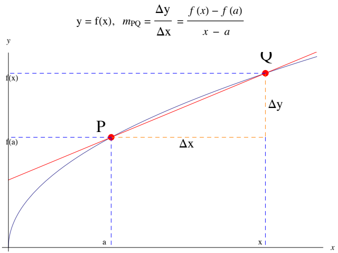

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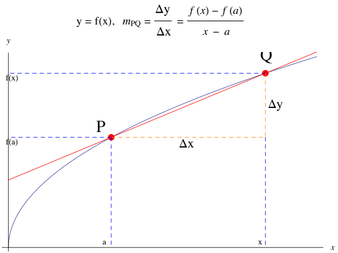

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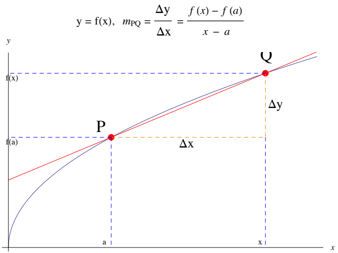

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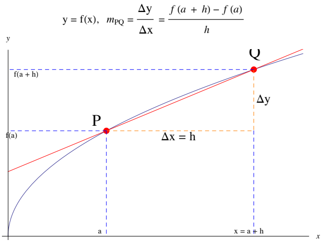
- The equation of the tangent to the curve at $P(1, 1)$ is $(y - 1) = m(x - 1)$ or $(y - 1) = \frac{1}{2}(x - 1)$ or $y = \frac{1}{2}x + \frac{1}{2}$.

Alternative form of limit



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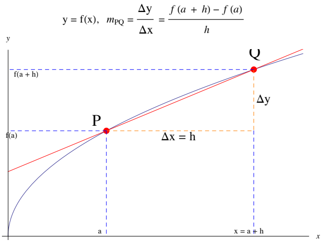

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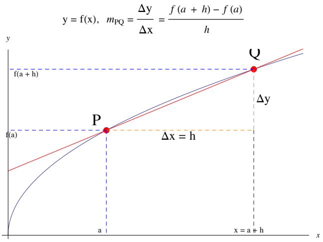

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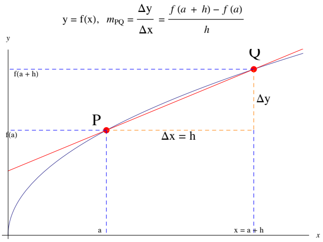

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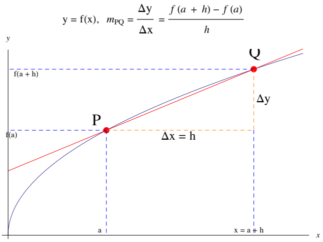

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- $= \lim_{h \to 0}(h+7) = 7$
- The tangent line at $(1, 6)$ is given by $y - 6 = 7(x - 1)$ or $y = 7x - 1$.

Derivative of a function at a point a

Definition When $f(x)$ is defined in an open interval containing a, the **derivative** of the function f at the number a is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

if the limit exists.

Note The slope of the tangent line to the graph $y = f(x)$ at the point $(a, f(a))$ is the derivative of f at a, $f'(a)$. (When $f'(a)$ exists it is sometimes referred to as the slope of the curve $y = f(x)$ at $x = a$.)

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- When $a = -1$, we get $f'(-1) = 2(-1) + 5 = 3$.

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(since the derivative of the function at a is the slope of the tangent at a.)
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- When the derivative of a function f at a is negative, the function is decreasing.
- When the absolute value of the derivative is small, the function is changing slowly (a small change in the value of x leads to a small change in the value of $f(x)$).

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Note Since the derivative of $f(x)$ at any point on the curve (if it exists) measures the slope of the curve or the slope of a tangent, our intuition allows us to make the following statements, which can be verified algebraically:

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- We will explore this in more detail next day.

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- $f(x) = x^4 + x$ and $a = 1$ works.

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If an object moves in a straight line, the displacement from the origin at time t is given by the **position function** $s = f(t)$, where s is the displacement of the object from the origin at time t.
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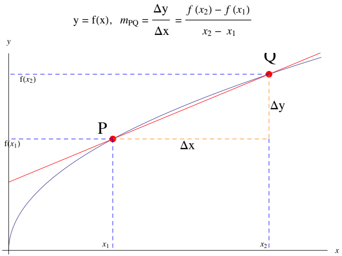
- Instantaneous velocity at $t = 1$ is -22 ft/s

Different Notation, Rates of change, Δx, Δy



With the above notation, $\Delta x = x_2 - x_1$ and $\Delta y = f(x_2) - f(x_1)$. Slope of the secant PQ = the difference quotient = $\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$ = **the average rate of change of y with respect to x.**

The **instantaneous rate of change of y with respect to x**, when $x = x_1$ is $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad (= f'(x_1))$.

In economics, the instantaneous rate of change of the cost function (revenue function) is called the **Marginal Cost** (Marginal Revenue).